Green's Functions in Robertson-Walker-Space*

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The wave equations for the metric of the cosmological standard model and their solutions in form of Green's functions are given.

The behaviour of electromagnetic radiation under cosmological conditions has been studied from different viewpoints for some time (cf. e.g. [1–6]). Such investigations are of increasing interest for an approach to the problem of a possible link between the electrodynamical and cosmological arrows of time. It is therefore useful to have at hand the Green's functions for the wave equation in the metric of the cosmological standard model (the Robertson-Walker-metric).

In a curved space-time, the inhomogeneous wave equation for a Green's function G takes on the following form [7]:

$$(\Box + \tfrac{1}{6}\hat{R})\, G = \frac{\delta(\eta-\eta_0)\,\delta(\varrho)}{R(\eta_0)\, R^3(\eta)\, \varrho^2}\,, \tag{1}$$

where η is a time variable defined below. Here $(\Box + \frac{1}{6}\hat{R})$ is the conformal version of the Beltrami operator composed of the general covariant d'Alembertian

$$\Box = \frac{1}{\sqrt{-g}}\,\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\; g^{\mu\nu}\,\frac{\partial}{\partial x^\nu}\right)$$

and the curvature invariant $\hat{R} = g^{\mu\varrho}\, g^{\nu\sigma}\, R_{\mu\nu\varrho\sigma}$ of the given space-time. $R_{\mu\nu\varrho\sigma}$ are the components of the Riemannian curvature tensor and $g = \det_{\mu\nu} g_{\mu\nu}$. The denominator at the right hand side of (1) is another Jacobian besides $\sqrt{-g}$.

The Robertson-Walker line element is commonly written in the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R^2(t)\left(\frac{\mathrm{d}\varrho^2}{1-\varkappa\varrho^2} + \varrho^2\,\mathrm{d}\Omega^2\right) \tag{2}$$

with the standard angular part $\mathrm{d}\Omega^2 = \mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2$. Here, $R(t)$ is a scale length which represents the change of all linear dimensions with time (e.g. the distance between two clusters of galaxies in arbitrary units). Furthermore, $K = \varkappa/R^2$ is the Gaussian curvature with $\varkappa = \operatorname{sgn} K$. In the case of positive curvature, $K>0$, the factor R is the usual world radius. The line element (2) is given in comoving coordinates where the cosmic medium is commonly approximated as dust, fluid, or radiation.

The Robertson-Walker-metric may be transformed into a form which is analytical in $\varkappa$ too. Thus we define [8, 9]

$$\varrho = \sin_\varkappa \psi := \begin{cases} \sin\psi & \varkappa = +1\,, \\ \psi \quad \text{for} & \varkappa = 0\,, \\ \sinh\psi & \varkappa = -1\,. \end{cases}$$

By introducing the new time variable

$$\eta = \int_{t_0}^{t} \frac{\mathrm{d}t'}{R(t')}$$

one can rewrite (2) as

$$\mathrm{d}s^2 = K^{-1}(\eta)\,[\mathrm{d}\eta^2 - (\mathrm{d}\psi^2 + \sin^2_\varkappa\psi\,\mathrm{d}\Omega^2)]\,.$$

The increment $\mathrm{d}\eta = \mathrm{d}t/R$ represents the distance in radians of a photon traveling on a three-dimensional spacelike hypersurface of radius R during the time $\mathrm{d}t$.

In the given line element we have to distinguish between three different cases:

a) For $K<0$ we have a (hyperbolic) open universe ($\varkappa = -1$);
b) for $K=0$ the space-part of space-time is euclidean (Einstein-de Sitter-universe; for $R(\eta) = \text{const}$ we recover the ordinary Minkowski space-time);
c) for $K>0$ we get a (spherical or elliptic) closed universe ($\varkappa = +1$).

The combined wave equation for all three cases reads:

$$\left[\frac{2R'}{R^3}\,\frac{\partial}{\partial\eta} + \frac{1}{R^2}\,\frac{\partial}{\partial\eta^2} - \frac{2}{R^2\cot_\varkappa\psi}\,\frac{\partial}{\partial\psi} - \frac{1}{R^2}\,\frac{\partial^2}{\partial\psi^2} + \frac{R''+\varkappa R}{R^3}\right] G_\varkappa(\eta,\psi) = \frac{\delta(\eta-\eta_0)\,\delta(\psi)}{R(\eta_0)\,R^3(\eta)\,\sin^2_\varkappa\psi} \tag{3}$$

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with the definitions $R' := \mathrm{d}R/\mathrm{d}\eta$, $\cos_\varkappa \psi := \sin'_\varkappa \psi$ and $\cot_\varkappa \psi := 1/\tan_\varkappa \psi = \cos_\varkappa \psi/\sin_\varkappa \psi$. The last term of the operator in (3) contains the explicit form of the curvature invariant $\hat{R} = 6\,(R'' + \varkappa R)/R^3$.

As may easily be checked, the retarded (−) and advanced (+) solutions for the three cases $\varkappa \,\varepsilon\, \{1, 0, -1\}$ are

$$G^{\pm}_{\varkappa}(\eta, \eta_0, \psi) = \frac{\delta(\eta - \eta_0 \pm \psi)}{2\,R(\eta_0)\,R(\eta)\sin_\varkappa \psi}\,. \tag{4}$$

In the Minkowski case ($\varkappa = 0$ and $R' = 0$) this becomes $r^{-1}\,\delta(t - t_0 \pm r)$ with $r = R\,\psi$, as expected.

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